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THE THERMOELASTICITY OF A MOVING PUNCH WHEN THE HEAT RELEASE FROM FRICTION IS TAKEN INTO ACCOUNT†

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The contact problem of the motion of a punch taking into account the heat released due to friction between the punch and a very thick elastic strip is considered. The contact problem of the theory of elasticity regarding a moving punch was previously investigated both ignoring [1, 2] and taking into account [3, 4] the forces of friction and the heat release. Here, unlike [3, 4], when solving the free quasi-stationary problem of thermoelasticity in a moving system of coordinates a relation is proposed between the coefficient of friction and the temperature. Particular attention is given to the problem of the possibility of a thermal explosion [5–11] or a sharp change (a bifurcation) in the contact temperatures. It is shown that a loss in the quasi-stationary thermoelastic stability occurs if the coefficient of friction increases linearly with the temperature. The proposed model can explain, to a first approximation, the avalanche-type wear of different moving components, for example, thin piston rings, due to their overheating.

1. Suppose an absolutely rigid punch with a flat base of width $2a$ (see Fig. 1) moves with constant velocity υ in the direction of the x axis over the upper boundary $y=h$ of an elastic strip of thickness h. The lower boundary of the strip $y=0$ lies without friction on an undeformed base. We will solve the problem in a moving system of coordinates $x'=x-\upsilon t$, $y'=y$, connected with the punch (the primes will henceforth be omitted). The punch is pressed against the strip with a force ρ per unit length of the punch, applied with an eccentricity e. Coulomb friction forces $\tau_{xy}=kq$, where $q=q(x)=-\sigma_y(y=h,\ |x|\leqslant a)$ is the contact pressure, occur in the region of contact between the punch and the strip.

Due to friction in the contact area a quantity of heat

$$Q=\upsilon\tau_{xy} \tag{1.1}$$

is released in unit time per unit area [7], which leads to heating of the surface of the strip, and also of the whole punch up to a temperature $T_*(x)=T(x,\ h)\ (|x|\ll a)$, which exceeds the temperature of the lower boundary of the strip $T_0=0$. We will assume here that on the lower boundary of the strip, and also outside the region of contact, the temperature remains that of the surrounding medium, which we will also take as the origin for the temperature readings. Hence, heat flow occurs through the strip, which, for $y=h$ is equal to [12]

$$Q=\lambda_*\partial T/\partial y \tag{1.2}$$

where λ_* is the thermal conductivity of the material of the elastic strip.

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Assuming that the heat conduction process is quasi-stationary with respect to a moving system of coordinates [13], we obtain the following heat-conduction equation

$$\Delta T + 2\omega \partial T / \partial x = 0, \quad \omega = \upsilon / 2a_* \tag{1.3}$$

where a_* is the thermal diffusivity of the material of the strip, with the following boundary conditions

$$y = 0: T = 0$$

$$y = h: \; T = 0, \; |x| > a; \; \partial T / \partial y = Q / \lambda_*, \; |x| \leq a \tag{1.4}$$

By searching for a solution of boundary-value problem (1.3), (1.4) in the form

$$T(x, y) = \frac{e^{-\omega x}}{2\pi} \int_{-\infty}^{\infty} U(\alpha) \frac{\operatorname{sh}\sqrt{\alpha^2 + \omega^2}\, y}{\operatorname{sh}\sqrt{\alpha^2 + \omega^2}\, h} e^{-i\alpha x} d\alpha$$

$$U(\alpha) = \int_{-a}^{a} u(x) e^{i\alpha x} dx, \quad u(x) = e^{\omega x} T_*(x) \tag{1.5}$$

assuming the function $u(x)$ to be continuous at the points $x = \pm a$, we obtain the following equation connecting the functions $u(x)$ and $Q(x)$

$$\int_{-a}^{a} u'(\xi) K_0(\xi - x) d\xi = -\pi e^{\omega x} Q(x) / \lambda_*, \quad |x| \leq a$$

$$K_0(t) \int_0^{\infty} \frac{L_0(\alpha)}{\alpha} \sin \alpha t d\alpha, \quad L_0(\alpha) = \sqrt{\alpha^2 + \omega^2} \operatorname{cth} \sqrt{\alpha^2 + \omega^2}\, h \tag{1.6}$$

To determine the stress–strain state of the strip (plane strain) we will use the Lamé–Neumann equations of linear uncoupled thermoelasticity [12] in a moving system of coordinates, neglecting the inertial terms. We will assume that the velocity υ is less than the velocities c_1, c_2 and c_p of the longitudinal, transverse and Rayleigh waves in the elastic strip, respectively, i.e. $\upsilon < c_p < c_2 < c_1$, where $c_1^2 = 2G(1-\nu)[\rho(1-2\nu)]^{-1}$, $c_2^2 = G\rho^{-1}$, and ρ, G and ν are the density, shear modulus and Poisson's ratio of the material of the strip, respectively. We will seek a solution of the Lamé–Neumann equations in the form [2]

$$u_x = \partial \Phi / \partial x + \partial \Psi / \partial y, \quad u_y = \partial \Phi / \partial y - \partial \Psi / \partial x \tag{1.7}$$

Then, neglecting the effect of the friction forces on the normal displacements under the punch [6], we have the following boundary-value problem

$$\begin{aligned}
&\Box_1^2 \Phi = \beta T, \quad \Box_2^2 \Psi = 0 \\
&y = 0: \; u_y = \tau_{xy} = 0 \\
&y = h: \; \sigma_y = 0, \; |x| > a; \; \sigma_y = -q(x), \; |x| \leq a; \; \tau_{xy} = 0 \\
&\Box_n^2 = \varepsilon_n^2 \partial^2 / \partial x^2 + \partial^2 / \partial y^2, \; \varepsilon_n^2 = 1 - \upsilon^2 / c_n^2, \; n = 1, 2
\end{aligned} \tag{1.8}$$

where $\beta = \alpha_*(1+\nu)/(1-\nu)$, and α_* is the coefficient of linear expansion of the material of the strip.

It follows from Hooke's law and Eqs (1.8) that

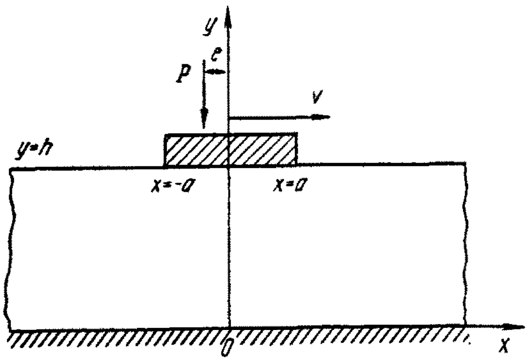

Fig. 1.

$$\frac{\sigma_y}{G}=-(1+\varepsilon_2^2)\frac{\partial^2\Phi}{\partial x^2}-2\frac{\partial^2\Psi}{\partial x\partial y},\quad \frac{\tau_{xy}}{G}=2\frac{\partial^2\Phi}{\partial x\partial y}-(1+\varepsilon_2^2)\frac{\partial^2\Psi}{\partial x^2} \tag{1.9}$$

We will represent the solution of boundary-value problem (1.8), (1.9) in the form of Fourier integrals

$$\Phi(x,\ y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\varphi_0(\alpha,\ y)+\beta e^{-\omega x}\varphi_1(\alpha,\ y)]e^{-i\alpha x}d\alpha$$

$$\Psi(x,\ y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_0(\alpha,\ y)e^{-i\alpha x}d\alpha \tag{1.10}$$

From the boundary conditions (1.8) with $y=0$ we obtain that

$$\varphi_1(\alpha,y)=\frac{U(\alpha)}{\kappa_2\operatorname{sh}\kappa_1 h}\frac{\kappa_2\operatorname{sh}\kappa_1 y-\kappa_1\operatorname{sh}\kappa_2 y}{\kappa_1^2-\kappa_2^2}$$

$$\kappa_1=\sqrt{\alpha^2+\omega^2},\quad \kappa_2=\varepsilon_1(\alpha-i\omega) \tag{1.11}$$

$$\varphi_0(\alpha,\ y)=A(\alpha)\operatorname{ch}(\varepsilon_1\alpha y),\quad \psi_0(\alpha,\ y)=B(\alpha)\operatorname{sh}(\varepsilon_2\alpha y)$$

By determining the unknown functions $A(\alpha)$, $B(\alpha)$ from the boundary conditions (1.8) with $y=h$ and shifting the contour of integration in the integral containing the function $\varphi_1(\alpha,\ y)$ (using the analyticity of the corresponding functions in the strip $-\omega<\operatorname{Im}\alpha<0$), we obtain an integral equation of the form ($u_y=-\delta$, $y=h$, $|x|\leqslant a$)

$$\frac{1}{G}\int_{-a}^{a}q(\xi)K_1(\xi-x)d\xi-\alpha_*\int_{-a}^{a}u(\xi)e^{-\omega\xi}K_2(\xi-x)d\xi=\pi\delta,\ |x|\leqslant a$$

$$K_n(t)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{L_n(\alpha)}{\alpha}e^{i\alpha t}d\alpha,\ n=1,\ 2 \tag{1.12}$$

$$L_1(\alpha)=\frac{\gamma_2\operatorname{th}(\varepsilon_1\alpha h)\operatorname{th}(\varepsilon_2\alpha h)}{\operatorname{th}(\varepsilon_1\alpha h)-\gamma_1\operatorname{th}(\varepsilon_2\alpha h)},\quad \gamma_1=\frac{(1+\varepsilon_2^2)^2}{4\varepsilon_1\varepsilon_2},\quad \gamma_2=\frac{1-\varepsilon_2^2}{4\varepsilon_2}$$

$$L_2(\alpha)=\frac{\beta(1+\varepsilon_2^2)\gamma_2\operatorname{th}(\varepsilon_2\alpha h)}{\varepsilon_2(\alpha(1-\varepsilon_1^2)+i2\omega)\alpha_*}\frac{\kappa[\operatorname{cth}(\kappa h)-\operatorname{cosech}(\kappa h)\operatorname{sech}(\varepsilon_1\alpha h)]-\varepsilon_1\alpha\operatorname{th}(\varepsilon_1\alpha h)}{\operatorname{th}(\varepsilon_1\alpha h)-\gamma_1\operatorname{th}(\varepsilon_2\alpha h)} \tag{1.13}$$

$$\kappa=\sqrt{\alpha(\alpha+2i\omega)}$$

When $\beta=0$, integral equation (1.12), (1.13) is identical with that obtained previously in [2, p. 289]. It can be shown that when $\upsilon<c_p$ the denominator in (1.13) for $L_{1,2}(\alpha)$ is positive and regular for all $\alpha>0$, since the velocity of the Rayleigh wave $\upsilon<c_p$ corresponds to the value $\gamma_1=1$ [2].

Taking the relation $Q(x)=k\upsilon q(x)$ ($|x|\leq a$) into account, we have a system of two integral equations (1.6) and (1.12) for determining the functions $u(x)$ and $q(x)$. Moreover, the following two integral conditions of equilibrium of the punch must obviously be satisfied

$$\int_{-a}^{a} q(x)dx = P, \quad \int_{-a}^{a} xq(x)dx = Pe \tag{1.14}$$

2. We will assume that the coefficient of friction depends linearly on the contact temperature $T_*(x)$ ($|x|\leq a$), i.e. taking (1.5) into account we have

$$k = k(T_*) = k_1 + k_2\beta e^{-\omega x}u(x) \tag{2.1}$$

where k_1 and k_2 are certain constants which depend on the materials of the rubbing pair (the punch and the strip).

Then, in new dimensionless notation we have

$$\begin{gathered} x'=x/a,\ \xi'=\xi/a,\ \omega'=\omega h,\ \delta'=\delta/a,\ \lambda=h/a \\ q'(x')=q(x)/G,\ u_0(x')=\alpha_* u(x),\ \alpha'=\alpha h,\ \kappa'=\kappa h \\ L_n'(\alpha)=L_n(\alpha')\ (n=0,\ 1,\ 2),\ \varepsilon=\alpha_*\upsilon Ga/\lambda_* \\ k_1'=k_1\varepsilon,\ k_2'=k_2\varepsilon\beta/\alpha_*,\ P'=P/Ga,\ e'=e/a \end{gathered} \tag{2.2}$$

and we can write the integral equations (1.6), (1.12) and (1.13) in the form (omitting the primes in (2.2))

$$\frac{1}{\lambda}\int_{-1}^{1} u_0'(\xi)K_0\left(\frac{\xi-x}{\lambda}\right)d\xi = -\pi(k_1e^{\omega x/\lambda}+k_2u_0(x))q(x),\ (|x|\leq 1) \tag{2.3}$$

$$\int_{-1}^{1} q(\xi)K_1\left(\frac{\xi-x}{\lambda}\right)d\xi - \int_{-1}^{1} u_0(\xi)e^{-\omega\xi/\lambda}K_2\left(\frac{\xi-x}{\lambda}\right)d\xi = \pi\delta,\ (|x|\leq 1) \tag{2.4}$$

We will investigate the system of non-linear equations (2.3) and (2.4) asymptotically as $\lambda\to\infty$, i.e. for the case of a relatively thick strip ($h/a\gg 1$). Using the well-known integrals [2]

$$\int_0^\infty \frac{e^{-u}-\cos ut}{u}du = \ln|t|,\quad \int_0^\infty \sin ut\,du = \frac{1}{t} \tag{2.5}$$

we can separate the principal singular terms in the kernels $K_n(t)$ ($n=0,\ 1,\ 2$) of Eqs (2.3) and (2.4). After differentiating Eq. (2.4) with respect to x, we can represent the system (2.3) and (2.4) as $\lambda\to\infty$ in the following form

$$\int_{-1}^{1} \frac{u_0'(\xi)}{\xi-x}d\xi = -\pi(k_1+k_2u_0(x))q(x),\ (|x|\leq 1) \tag{2.6}$$

$$\int_{-1}^{1} \frac{q(\xi)+C_*u_0(\xi)}{\xi-x}d\xi = 0,\quad C_* = \frac{\beta(1+\varepsilon_2^2)}{\alpha_*(1-\varepsilon_1^2)} \tag{2.7}$$

From (2.7) we obtain that [2]

$$q(x)+C_*u_0(x)=R/\sqrt{1-x^2} \tag{2.8}$$

where the constant $R>0$ can obviously be expressed in terms of the force P and the mean contact temperature

$$u_*=\int_{-1}^{1}u_0(x)dx,\ R=(P+C_*u_*)/\pi \tag{2.9}$$

We expressed the function $q(x)$ in terms of $u_0(x)$ using (2.8) and substitute it into Eq. (2.6). Now regarding Eq. (2.6) as a Prandtl-type integro-differential equation in the function $u_0(x)$ and employing the well-known method from [2, p. 206], we can reduce it to the following equivalent Hammerstein integral equation

$$u_0(x)=\frac{1}{2\pi}\int_{-1}^{1}F(\xi,\ x)\left\{-k_2C_*u_0^2(\xi)+\left[\frac{k_2P}{\sqrt{1-\xi^2}}-k_1C_*\right]u_0(\xi)+\frac{k_1R}{\sqrt{1-\xi^2}}\right\}d\xi, \tag{2.10}$$

$|x|\leqslant 1$

$$F(\xi,\ x)=\ln\frac{1-\xi x+\sqrt{(1-\xi^2)(1-x^2)}}{1-\xi x-\sqrt{(1-\xi^2)(1-x^2)}} \tag{2.11}$$

We will investigate the possibility of the "branching" of the solutions from the known solution $u_*(x)$ of Eq. (2.10) (a bifurcation point) [14]. To do this we will replace the required function in (2.10) in accordance with the formula $u_0(x)=y(x)+u_*(x)$ and we will take the Frechet differential when $y(x)\equiv 0$ of the Hammerstein operator in the integral equation of the function $y(x)$. We obtain the following linear homogeneous integral equation

$$h(x)-\frac{k_2}{2\pi}\int_{-1}^{1}F(\xi,\ x)\gamma(\xi)h(\xi)d\xi=0,\ |x|\leqslant 1 \tag{2.12}$$

$$\gamma(\xi)=\frac{R}{\sqrt{1-\xi^2}}-2C_*u_*(\xi)-C_*\frac{k_1}{k_2} \tag{2.13}$$

Suppose $k_1=\mu k_2$, $\mu=\text{const}$, and the force P is so large that the condition $\gamma(\xi)>0(|\xi|<1)$ is satisfied for the function (2.13). Then Eq. (2.12) can be considered in the space $L_\gamma^2(-1,\ 1)$ with weight $\gamma(\xi)$, where it is an integral equation with a Hilbert–Schmidt kernel, which, moreover, is a positive-definite kernel. Consequently, all the characteristic numbers $k_2=\lambda_n$ $(n=1,\ 2,\ \ldots)$, corresponding to Eq. (2.12) are positive. Each such odd-multiple (in particular, simple) number λ_n will be the required bifurcation point [14].

We will now approximate the function $q(x)$ by the expression $P/(\pi\sqrt{(1-x^2)})$, as was done in [6], i.e. we will put $C_*=0$. Taking into account the spectral relation [15]

$$\frac{1}{2\pi}\int_{-1}^{1}F(\xi,\ x)U_{n-1}(x)dx=\sqrt{1-\xi^2}\ \frac{U_{n-1}(\xi)}{n}\ (n=1,\ 2,\ldots) \tag{2.14}$$

where $U_{n-1}(x)$ are Chebyshev polynomials of the second kind, the solution of Eq. (2.10) with $C_*=0$ can be represented in the form

$$u_0(x)=\sqrt{1-x^2}\ \sum_{n=1}^{\infty}b_nU_{n-1}(x) \tag{2.15}$$

The free term in (2.10) can also be expanded in a series in Chebyshev polynomials

$$\frac{k_1 P}{2\pi^2} \int_{-1}^{1} \frac{F(\xi, x)}{\sqrt{1-\xi^2}} d\xi = \sqrt{1-x^2} \sum_{n=1}^{\infty} a_n U_{n-1}(x) \tag{2.16}$$

where the coefficients a_n can be found from the condition

$$\int_{-1}^{1} U_{n-1}(\xi) U_{m-1}(\xi) \sqrt{1-\xi^2} d\xi = \begin{cases} 0, & n \neq m \\ \pi/2 & n = m \end{cases} \quad (n, \ m = 1, \ 2, \ldots)$$

The unknown constants b_n in expansion (2.15) can be found from the formulae

$$b_n = \pi n a_n / (\pi n - k_2 P) \quad (n = 1, \ 2, \ldots) \tag{2.17}$$

It follows from (2.17) that the existence of a quasi-stationary heat-conduction mode becomes impossible when $k_2 P = \pi n$ $(n = 1, 2, \ldots)$. Hence, changing to dimensional quantities, we obtain the critical velocity of motion of the punch

$$\upsilon_n = \pi n \frac{1-\nu}{1+\nu} \frac{\lambda_*}{P \alpha_* k_2} \quad (n = 1, \ 2, \ldots) \tag{2.18}$$

Hence, a thermal explosion can only occur when $k_2 > 0$, i.e. when the coefficient of friction increases linearly with the temperature. A similar result was obtained when considering other problems in [6–9]. It can be seen from (2.18) that the worse the thermal conductivity of the elastic strip and the larger the force pressing down on the moving punch, the lower the threshold of the first critical velocity υ_1.

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